SPLITTING HOMOMORPHISMS AND THE GEOMETRIZATION CONJECTURE

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ABSTRACT. This paper gives an algebraic conjecture which is shown to be equivalent to Thurston's Geometrization Conjecture for closed, orientable 3-manifolds. It generalizes the Stallings-Jaco theorem which established a similar result for the Poincaré Conjecture. The paper also gives two other algebraic conjectures; one is equivalent to the finite fundamental group case of the Geometrization Conjecture, and the other is equivalent to the union of the Geometrization Conjecture and Thurston's Virtual Bundle Conjecture.

1. Introduction

The Poincaré Conjecture states that every closed, simply connected 3-manifold is homeomorphic to S^3. Stallings [27] and Jaco [11] have shown that the Poincaré Conjecture is equivalent to a purely algebraic conjecture. Let S be a closed, orientable surface of genus g and F_1 and F_2 free groups of rank g. A homomorphism $\varphi = \varphi_1 \times \varphi_2 : \pi_1(S) \to F_1 \times F_2$ is called a *splitting homomorphism* of genus g if φ_1 and φ_2 are onto. It has an *essential factorization* through a free product if $\varphi = \theta \circ \psi$, where $\psi : \pi_1(S) \to A * B$, $\theta : A * B \to F_1 \times F_2$, and $\operatorname{im} \psi$ is not conjugate into A or B.

Theorem 1.1 (Stallings-Jaco). *The Poincaré Conjecture is true if and only if every splitting epimorphism of genus $g > 1$ has an essential factorization.* □

Thurston's Geometrization Conjecture [28, Conjecture 1.1] is equivalent to the statement that each prime connected summand of a closed, connected, orientable 3-manifold either is Seifert fibered, is hyperbolic, or contains an incompressible torus. (See [24].) In particular, it implies that a closed, connected, orientable 3-manifold with finite fundamental group must be Seifert fibered. Since a closed, simply connected Seifert fibered space must be homeomorphic to S^3, the Poincaré Conjecture is a special case of the Geometrization Conjecture. The goal of this paper is to generalize the Stallings-Jaco theorem to the setting of the Geometrization Conjecture.

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The special case in which the fundamental group is finite is the closest analogue of the Stallings-Jaco theorem. Denote by $[G : H]$ the index of the subgroup H of the group G.

Theorem 1.2. *The Geometrization Conjecture is true for closed, connected, orientable 3-manifolds with finite fundamental group if and only if every splitting homomorphism φ of genus $g > 2$ such that $[F_1 \times F_2 : \operatorname{im}\varphi] < \infty$ has an essential factorization.*

For the general case we have the following result.

Theorem 1.3. *The Geometrization Conjecture is true if and only if for every splitting homomorphism φ of genus $g > 2$ either*

(1) *φ has an essential factorization, or*
(2) *$\pi_1(S)/\ker\varphi_1 \ker\varphi_2$ either*
  (a) *contains a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup, or*
  (b) *is isomorphic to a discrete, non-trivial, torsion-free subgroup of $SL(2, \mathbf{C})$.*

Thurston has also conjectured [28, p. 380] that every closed, connected, hyperbolic 3-manifold has a finite sheeted covering space which is a surface bundle over S^1. Combining this "Virtual Bundle Conjecture" with the Geometrization Conjecture gives an "Extended Geometrization Conjecture" which is also equivalent to an algebraic conjecture. Define a subgroup H of a group G to be *good* if it is finitely generated and non-trivial and $N(H)/H$ has an element of infinite order, where $N(H)$ is the normalizer $\{g \in G \mid gHg^{-1} = H\}$ of H in G.

Theorem 1.4. *The Extended Geometrization Conjecture is true if and only if for every splitting homomorphism φ of genus $g > 1$ either*

(1) *φ has an essential factorization, or*
(2) *$\pi_1(S)/\ker\varphi_1 \ker\varphi_2$ has a good subgroup, or*
(3) *$[F_1 \times F_2 : \operatorname{im}\varphi] < \infty$, and $g = 2$.*

The paper is organized as follows. Section 2 quotes some background lemmas. Section 3 proves some algebraic results. Sections 4, 5, and 6 prove Theorems 1.2, 1.3, and 1.4, respectively. Section 7 gives an alternative proof of Thurston's observation [28, p. 380] (see also Culler and Shalen [5, Theorem 4.2.1] and Gabai [7]) that closed, orientable, virtually hyperbolic 3-manifolds are homotopy hyperbolic.

Unless the contrary is evident all manifolds under consideration are assumed to be connected.

2. Heegaard splittings and splitting homomorphisms

Recall that a *Heegaard splitting* of a closed, orientable 3-manifold M is a pair (M, S), where S is a closed, orientable surface in M such that $M - S$ has two components, and the closures of these components are cubes with handles, which we

denote by V_1 and V_2. The *genus* of the splitting is the genus of S. The splitting is *reducible* if there is a 2-sphere Σ in M which is in general position with respect to S such that $S \cap \Sigma$ is a simple closed curve which does not bound a disk on S. Recall also that M is said to be *reducible* if it contains a 2-sphere which does not bound a 3-ball.

Lemma 2.1 (Haken). *Every Heegaard splitting of a reducible 3-manifold is reducible.*

Proof. See [9, p. 84]. See also [12, Theorem II.7]. □

It follows that if (M, S) is a genus g Heegaard splitting of a reducible 3-manifold M, then either $g = 1$ and M is homeomorphic to $S^1 \times S^2$ or $g > 1$ and M can be expressed as a connected sum of 3-manifolds having Heegaard splittings of lower genera. See [10, Lemma 3.8].

Two splitting homomorphisms $\varphi : \pi_1(S) \to F_1 \times F_2$ and $\varphi' : \pi_1(S') \to F_1' \times F_2'$ are *equivalent* if there are isomorphisms $\sigma_i : F_i \to F_i'$ and $\tau : \pi_1(S) \to \pi_1(S')$ such that $\sigma \circ \varphi = \varphi' \circ \tau$, where $\sigma = \sigma_1 \times \sigma_2$. Note that in this case φ has an essential factorization if and only if φ' does.

Every Heegaard splitting gives rise to a splitting homomorphism by choosing a basepoint on S, and, for $i = 1, 2$, letting $F_i = \pi_1(V_i)$ and letting φ_i be the induced homomorphism on fundamental groups. A splitting homomorphism is *realized* by a Heegaard splitting if it is equivalent to a splitting homomorphism of this type.

Lemma 2.2 (Jaco). *Every splitting homomorphism can be realized by a Heegaard splitting of some 3-manifold.*

Proof. This is Theorem 5.2 of [11]. □

Lemma 2.3 (Stallings-Jaco). *Suppose $g > 1$. Then (M, S) is reducible if and only if the associated splitting homomorphism has an essential factorization.*

Proof. Sufficiency is due to Stallings [27, Theorem 2] and necessity to Jaco [11, pp. 377–378]. □

Lemma 2.4 (Stallings). *$\pi_1(M)$ is isomorphic to $\pi_1(S)/\ker\varphi_1 \ker\varphi_2$.*

Proof. See [27, p. 85]. See also [14, p. 128]. □

We briefly sketch how these ingredients give the Stallings-Jaco theorem. Stallings showed that $\pi_1(M)$ is trivial if and only if φ is onto [27, Theorem 1]. (See also Lemma 3.1 below.) Thus if every splitting epimorphism of genus greater than one has an essential factorization then every Heegaard splitting of genus greater than one of a homotopy 3-sphere is reducible, and so one can express it as a connected sum of homotopy 3-spheres with genus one Heegaard splittings, which must be homeomorphic to S^3. Jaco took an arbitrary splitting epimorphism of genus greater than one and realized it by a Heegaard splitting of a homotopy 3-sphere. If it is homeomorphic

to S^3, then by a result of Waldhausen [29] the splitting is reducible, and hence the splitting epimorphism has an essential factorization.

We finally remark that the Geometrization Conjecture is well known to hold for closed, orientable 3-manifolds with Heegaard splittings of genus at most two. For genus at most one this is classical. For genus two, such manifolds admit involutions with 1-manifolds as fixed point sets [1]. The result then follows from Thurston's Orbifold Theorem, which was announced in [28, p. 362], and has been given detailed proofs by Cooper, Hodgson, and Kerckhoff and also by Boileau and Porti [2] in the case of a cyclic group action with 1-dimensional fixed point set.

3. Some algebraic results

For a subgroup H of a group G let G/H denote the set of left cosets $[g] = gH$ of H in G.

Lemma 3.1. *Let φ be a splitting homomorphism. There is a bijection*

$$\Phi : \pi_1(M) \cong \pi_1(S)/\ker\varphi_1\ \ker\varphi_2 \to F_1 \times F_2 \,/\, \mathrm{im}\,\varphi$$

given by $\Phi([x]) = [(\varphi_1(x), 1)]$.

Proof. Φ is well defined: Suppose $y = xk_1k_2$, where $k_i \in \ker\varphi_i$. Then

$$\begin{aligned}
\Phi([y]) &= [(\varphi_1(xk_1k_2), 1)] \\
&= [(\varphi_1(x)\varphi_1(k_1)\varphi_1(k_2), 1)] \\
&= [(\varphi_1(x)\varphi_1(k_2), 1)] \\
&= [(\varphi_1(x)\varphi_1(k_2), \varphi_2(k_2))] \\
&= [(\varphi_1(x), 1)(\varphi_1(k_2), \varphi_2(k_2))] \\
&= [(\varphi_1(x), 1)\varphi(k_2)] \\
&= [(\varphi_1(x), 1)] \\
&= \Phi([x])
\end{aligned}$$

Φ is one to one: If $\Phi([x]) = \Phi([y])$, then for some $z \in \pi_1(S)$ one has

$$\begin{aligned}
(\varphi_1(x), 1) &= (\varphi_1(y), 1)\varphi(z) \\
&= (\varphi_1(y), 1)(\varphi_1(z), \varphi_2(z)) \\
&= (\varphi_1(y)\varphi_1(z), \varphi_2(z))
\end{aligned}$$

Thus $z \in \ker\varphi_2$, and $\varphi_1(x) = \varphi_1(yz)$. Let $k_1 = x(yz)^{-1}$. Then $k_1 \in \ker\varphi_1$, and $x = k_1yz = yk_1'z$ for some $k_1' \in \ker\varphi_1$ since this subgroup is normal in $\pi_1(S)$. Thus $[x] = [y]$.

Φ is onto: Let $(a, b) \in F_1 \times F_2$. Since the φ_i are onto we have $a = \varphi_1(x)$ and $b = \varphi_2(y)$ for some $x, y \in \pi_1(S)$. Thus

$$\begin{aligned} [(a,b)] &= [(\varphi_1(x), \varphi_2(y))] \\ &= [(\varphi_1(xy^{-1})\varphi_1(y), \varphi_2(y)] \\ &= [(\varphi_1(xy^{-1}), 1)(\varphi_1(y), \varphi_2(y))] \\ &= [(\varphi_1(xy^{-1}), 1)] \\ &= \Phi([xy^{-1}]) \end{aligned}$$

□

In general Φ need not be an isomorphism because $\operatorname{im}\varphi$ need not be normal in $F_1 \times F_2$, and so $F_1 \times F_2 \,/\, \operatorname{im}\varphi$ need not be a group. In fact, one has the following precise result. Let $Z(G)$ denote the center of the group G. Recall that $N(H)$ denotes the normalizer of the subgroup H of G.

Proposition 3.2. $\Phi(Z(\pi_1(M))) = N(\operatorname{im}\varphi)/\operatorname{im}\varphi$.

Proof. Suppose $[y] \in Z(\pi_1(M))$. Let $x \in \pi_1(S)$. Then $yxy^{-1}x^{-1} = k_1k_2$ for some $k_i \in \ker\varphi_i$, $i = 1, 2$. So $\varphi_1(yxy^{-1}) = \varphi_1(yxy^{-1}x^{-1}x) = \varphi_1(k_1k_2x) = \varphi_1(k_2x)$ and $\varphi_2(x) = \varphi_2(k_2x)$. Thus

$$(\varphi_1(y), 1)(\varphi_1(x), \varphi_2(x))(\varphi_1(y^{-1}), 1) = (\varphi_1(yxy^{-1}), \varphi_2(x)) = (\varphi_1(k_2x), \varphi_2(k_2x)).$$

Similarly $y^{-1}xyx^{-1} = k_1'k_2'$ implies that

$$(\varphi_1(y^{-1}), 1)(\varphi_1(x), \varphi_2(x))(\varphi_1(y), 1) = (\varphi_1(k_2'x), \varphi_2(k_2'x)).$$

Hence $(\varphi_1(y), 1) \in N(\operatorname{im}\varphi)$.

Now suppose that $(a, b) \in N(\operatorname{im}\varphi)$. From the proof that Φ is onto in Lemma 3.1 we may assume that $(a, b) = (\varphi_1(y), 1)$ for some $y \in \pi_1(S)$. Let $x \in \pi_1(S)$. Then $(\varphi_1(y), 1)(\varphi_1(x), \varphi_2(x))(\varphi_1(y^{-1}), 1) = (\varphi_1(z), \varphi_2(z))$ for some $z \in \pi_1(S)$. So $\varphi_1(yxy^{-1}) = \varphi_1(z)$ and $\varphi_2(x) = \varphi_2(z)$. Hence $yxy^{-1}z^{-1} = k_1 \in \ker\varphi_1$ and $zx^{-1} = k_2 \in \ker\varphi_2$. So $yxy^{-1}x^{-1} = k_1k_2 \in \ker\varphi_1 \ker\varphi_2$. Hence $[y] \in Z(\pi_1(M))$. □

Corollary 3.3. *$\operatorname{im}\varphi$ is normal in $F_1 \times F_2$ if and only if $\pi_1(M)$ is abelian.* □

4. The finite fundamental group case

Proof of Theorem 1.2. First assume that every splitting homomorphism φ with $g > 2$ and $[F_1 \times F_2 : \operatorname{im}\varphi] < \infty$ has an essential factorization. Let M be a closed, orientable 3-manifold with $\pi_1(M)$ finite. We may assume that M is irreducible. Let (M, S) be a Heegaard splitting of M of minimal genus g. Let φ be the associated splitting homomorphism. By Lemma 3.1 $[F_1 \times F_2 : \operatorname{im}\varphi] < \infty$, and so if g were greater than two, then φ would have an essential factorization, and hence (M, S) would be reducible. Since M is irreducible this would yield a Heegaard splitting of lower genus, contradicting the choice of g. Thus $g \leq 2$, and we are done.

Now assume that the Geometrization Conjecture holds in the finite fundamental group case. Let φ be a splitting homomorphism with $[F_1 \times F_2 : \operatorname{im} \varphi] < \infty$ and genus $g > 2$. Realize φ by a Heegaard splitting (M, S). Then by Lemma 3.1 $\pi_1(M)$ is finite. Suppose φ does not have an essential factorization. Then (M, S) is irreducible.

By the Geometrization Conjecture M is a Seifert fibered space. Since $\pi_1(M)$ is finite M has a Seifert fibration over a 2-sphere with at most three exceptional fibers. (See [10, Theorem 12.2] or [12, p. 92]. Note that it may have a Seifert fibration over a projective plane with one exceptional fiber, but then it also has a Seifert fibration of the given type.) If there were fewer than three exceptional fibers, then M would be S^3 or a lens space. But by results of Waldhausen [29] and of Bonahon and Otal [3] the irreducible Heegaard splittings of these spaces have, respectively, genus zero and one, contradicting our choice of g. Thus there are three exceptional fibers f_i of multiplicities $\alpha_i > 1$, $i = 1, 2, 3$. Moreover, up to ordering, $(\alpha_1, \alpha_2, \alpha_3)$ must be one of $(2, 2, \alpha_3)$, $\alpha_3 \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$.

We now recall two constructions for Heegaard splittings of closed, orientable Seifert fibered spaces over orientable base surfaces. For simplicity we restrict to the special case at hand. See [17] and [25] for the general case and a more detailed description.

First choose two of the three exceptional fibers. Join their image points in the base 2-sphere by an arc which misses the image point of the other exceptional fiber. Lift this arc to an arc in M joining the two chosen exceptional fibers. A regular neighborhood V of the resulting graph is a cube with two handles. It turns out that the closure of its complement is also a cube with handles, and so $(M, \partial V)$ is a genus two Heegaard splitting of M. It is called a *vertical* Heegaard splitting. We remark that in the general case all vertical Heegaard splittings have the same genus g_v.

Next choose one exceptional fiber f_i, and let N be a regular neighborhood of it. The closure M_0 of the complement of N is bundle over S^1 with fiber a surface F [10, Theorem 12.7], [12, Theorem VI.32]. Moreover, F is a branched covering space of the base surface of the Seifert fibration; the branch points are the images of the exceptional fibers and have branching indices equal to the indices of the exceptional fibers. Suppose ∂F is connected and has intersection number ± 1 with a meridian of the solid torus N. Let H be a regular neighborhood of F in M_0. Then H is a cube with handles whose genus is twice the genus of F. It turns out that the closure of the complement of H in M is also a cube with handles, and thus $(M, \partial H)$ is a Heegaard splitting of M. It is called a *horizontal* Heegaard splitting at f_i. Denote its genus by $g_h(f_i)$. Note that if either of the two conditions on ∂F is violated, then by definition M does not have a horizontal Heegaard splitting at f_i. Let d be the least common multiple of α_j and α_k, where f_j and f_k are the other two exceptional fibers.

Moriah and Schultens [17] have shown that every irreducible Heegaard splitting of a closed, orientable Seifert fibered space over an orientable base surface is either vertical or horizontal. Since $g > 2$ and $g_v = 2$ our splitting (M, S) must be horizontal.

Sedgwick [25] has determined precisely which vertical and horizontal Heegaard splittings are irreducible. In particular a horizontal splitting is irreducible if and only

if either $g_h(f_i) \leq g_v$ or $\alpha_i > d$. In our case the first condition is impossible, and the second condition holds only for $(2, 2, \alpha_3)$, where $\alpha_3 > 2$. But in this case M_0 is Seifert fibered over a disk with two exceptional fibers of index two and so must be a twisted I-bundle over a Klein bottle; it follows that the fiber F is an annulus, and so M does not have a horizontal Heegaard splitting at f_3.

Thus (M, S) must be reducible, and so φ must have an essential factorization. □

5. The general case

Proof of Theorem 1.3. Suppose the condition on the φ holds. Let M be a closed, orientable, irreducible 3-manifold and (M, S) a Heegaard splitting of minimal genus g. If $g \leq 2$, then the Geometrization Conjecture holds for M. So assume $g > 2$. If φ had an essential factorization, then (M, S) would be reducible, and, since M is irreducible M would have a Heegaard splitting of lower genus, contradicting the choice of g. So φ does not have an essential factorization, and we must be in case (2).

In case (2)(a) $\pi_1(M)$ has a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup. By Scott's version of the torus theorem [23] either M contains an incompressible torus, and we are done, or $\pi_1(M)$ contains a normal $\mathbf{Z}$ subgroup. In the latter case the proof of the Seifert fibered space conjecture by Casson and Jungreis [4] and by Gabai [6] gives that M is a Seifert fibered space, and again we are done.

In case (2)(b) $\pi_1(M)$ is isomorphic to a discrete, non-trivial, torsion-free subgroup of $SL(2, \mathbf{C})$. Since the kernel of the projection to $PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\{\pm I\}$ is a finite group this subgroup projects isomorphically to a discrete subgroup Γ of $PSL(2, \mathbf{C})$. A subgroup of $PSL(2, \mathbf{C})$ is discrete and torsion free if and only if its natural action on hyperbolic 3-space $\mathbf{H}^3$ is free (no non-trivial element has a fixed point) and discontinuous (each compact set meets only finitely many of its translates) [21, Theorems 8.2.1, 8.1.2, and 5.3.5]. Thus the quotient space $N = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold [21, Theorem 8.1.3]. Since M is closed and aspherical $H_3(N) \cong H_3(M) \cong \mathbf{Z}$, and so N is closed and orientable. By the topological rigidity of hyperbolic 3-manifolds, due to Gabai, Meyerhoff, and N. Thurston [8] we have that M and N are homeomorphic, and we are done.

Now suppose that the Geometrization Conjecture is true. Let φ be a splitting homomorphism of genus $g > 2$. Assume that φ has no essential factorization. Let (M, S) be a Heegaard splitting which realizes φ. Then (M, S) is irreducible, and hence M is irreducible. By Lemma 3.1 and Theorem 1.2 we may assume that $\pi_1(M)$ is infinite. Since M is irreducible, $\pi_1(M)$ is torsion-free [10, Corollary 9.9].

If M is hyperbolic, then $M = \mathbf{H}^3/\Gamma$ for some subgroup Γ of $PSL(2, \mathbf{C})$ acting freely and discontinuously on $\mathbf{H}^3$. Thus Γ is discrete and torsion free. By a result of Thurston Γ lifts isomorphically to a subgroup of $SL(2, \mathbf{C})$. (See Culler and Shalen [5, Proposition 3.1.1].)

If M contains an incompressible torus, then clearly $\pi_1(M)$ contains a $\mathbf{Z}\oplus\mathbf{Z}$ subgroup. If M is Seifert fibered, then $\pi_1(M)$ infinite implies that M has a covering space which is homeomorphic to an S^1 bundle over a closed, orientable surface F of positive genus [24, p. 438], and so again $\pi_1(M)$ contains a $\mathbf{Z}\oplus\mathbf{Z}$ subgroup. (See also [24, p. 477].) $\square$

6. The extended conjecture

Recall that a non-trivial subgroup H of a group G is *good* if it is finitely generated and $N(H)/H$ has an element of infinite order.

Lemma 6.1. *Let M be a closed, orientable, irreducible 3-manifold. Then $\pi_1(M)$ has a good subgroup if and only if either M has a finite sheeted covering space which is a surface bundle over S^1 or $\pi_1(M)$ contains a $\mathbf{Z}\oplus\mathbf{Z}$ subgroup.*

Proof. If M is covered by a bundle with fiber a surface F, then the image of $\pi_1(F)$ in $\pi_1(M)$ is a good subgroup. If $\pi_1(M)$ contains a $\mathbf{Z}\oplus\mathbf{Z}$ subgroup, then a summand is a good subgroup.

The converse follows from [19, Lemma 1]. For convenience we give the relevant portion of the proof of that result. Let H be a good subgroup and $\widetilde{M}$ the covering of M corresponding to H. Since the group of covering translations is isomorphic to $N(H)/H$ [14, Corollary 7.3] there is a covering translation of infinite order. Let M^* be the quotient of $\widetilde{M}$ by this covering translation. $\pi_1(M^*)$ has a normal subgroup which is isomorphic to H and has infinite cyclic quotient. The Scott compact core [22] C of M^* is a compact submanifold of M^* with $\pi_1(C)$ isomorphic to $\pi_1(M^*)$. Since M^* is irreducible [15] we may assume that C is irreducible. It then follows from the Stallings fibration theorem [26] that C is a surface bundle over S^1. If $C = M^*$, then we are done. If $C \neq M^*$, then ∂C consists of tori which are incompressible in M^* and so $\pi_1(M)$ has a $\mathbf{Z}\oplus\mathbf{Z}$ subgroup. $\square$

Proof of Theorem 1.4. Suppose the extended conjecture is true. If $g > 2$, then by the proof of Theorem 1.3 we reduce to the situation in which either $\pi_1(M)$ has a $\mathbf{Z}\oplus\mathbf{Z}$ subgroup or M is hyperbolic; in the latter case we apply the Virtual Bundle Conjecture and so conclude in both cases that $\pi_1(M)$ has a good subgroup. If $g = 2$, then a similar argument shows that either $\pi_1(M)$ has a good subgroup or M is a Seifert fibered space with $\pi_1(M)$ finite, in which case $[F_1 \times F_2 : \operatorname{im}\varphi] < \infty$.

Now assume that the conditions on the φ hold. Let M be a closed, orientable, irreducible 3-manifold. By arguments similar to those in the proof of Theorem 1.3 we reduce to the case that $\pi_1(M)$ has a good subgroup and does not have a $\mathbf{Z}\oplus\mathbf{Z}$ subgroup. Then by Lemma 6.1 M is finitely covered by a surface bundle. Note that this is the case if M is assumed to be hyperbolic [16, p. 52], [24, Corollary 4.6], and so the Virtual Bundle Conjecture holds. For the general situation note that by the fibered case of Thurston's hyperbolization theorem [28, Theorem 2.3], [20] the surface

bundle is hyperbolic. We conclude from the following result that the Geometrization Conjecture holds for M.

Lemma 6.2. *Let M be a closed, orientable 3-manifold which has a finite sheeted covering space M^* which is hyperbolic. Then M is hyperbolic.*

Proof. This is well known. It follows immediately from the topological rigidity of hyperbolic 3-manifolds [8] and the observation of Thurston that M is homotopy equivalent to a hyperbolic 3-manifold. See [5, Theorem 4.2.1] or the next section for a proof. □

This concludes the proof of Theorem 1.4. □

7. Virtually hyperbolic 3-manifolds

The following result was observed by Thurston [28, p. 380] to be a consequence of the Mostow rigidity theorem [18]. A proof was given by Culler and Shalen [5, Theorem 4.2.1]; a sketch of the proof has also been given by Gabai [7]. In this section we fill in some details of this sketch to give a proof which, though similar to that of Culler and Shalen, makes somewhat less explicit use of hyperbolic geometry.

Lemma 7.1 (Thurston)**.** *Let M be a closed, orientable 3-manifold which has a finite sheeted covering space which is hyperbolic. Then M is homotopy equivalent to a hyperbolic 3-manifold.*

Proof. We may assume that the covering is regular [13, Theorem 4.7].

As pointed out by Gabai, the covering translation in M^* corresponding to an element of $\pi_1(M)$ is by Mostow rigidity homotopic to a unique isometry. The lifts of these isometries to the universal cover $\mathbf{H}^3$ give a subgroup Γ of *Isom*$(\mathbf{H}^3)$. The quotient $N = \mathbf{H}^3/\Gamma$ is then the desired hyperbolic 3-manifold. To fill out this sketch one needs to verify that $\pi_1(M) \cong \Gamma$ and that Γ acts freely and discontinuously on $\mathbf{H}^3$. It will be convenient to first establish the following result.

Lemma 7.2. *Let M^* be a closed hyperbolic 3-manifold and $H : M^* \times I \to M^*$ a homotopy such that $H(x, 0) = H(x, 1) = x$ for all $x \in M^*$. Fix $y_0^* \in M^*$. Let $m(t) = H(y_0^*, t)$ for all $t \in I$. Then the class μ of m in $\pi_1(M^*, y_0^*)$ is trivial.*

Proof. Let $\lambda \in \pi_1(M^*, y_0^*)$ be represented by the loop $\ell(s)$. Then the map $G : I \times I \to M^*$ given by $G(s, t) = H(\ell(s), t)$ shows that $\mu\lambda = \lambda\mu$. Hence $\mu \in Z(\pi_1(M^*, y_0^*))$, which is trivial for a closed hyperbolic 3-manifold [16, p. 52]. □

Returning to the proof of Lemma 7.1, we have covering maps $\mathbf{H}^3 \xrightarrow{p} M^* \xrightarrow{q} M$. Choose a basepoint $\tilde{x}_0 \in \mathbf{H}^3$, and let $x_0^* = p(\tilde{x}_0)$ and $x_0 = q(x_0^*)$. Given $\alpha \in \pi_1(M, x_0)$, let α^* and $\tilde{\alpha}$ be path classes lifting α with $\alpha^*(0) = x_0^*$ and $\tilde{\alpha}(0) = \tilde{x}_0$, respectively. There is a covering translation f_0 of q such that $f_0(x_0^*) = \alpha^*(1)$ and a lifting $\tilde{f}_0$ of f_0 such that $\tilde{f}_0(\tilde{x}_0) = \tilde{\alpha}(1)$. By Mostow rigidity there is a unique isometry f_1 of M^*

which is homotopic to f_0. Let f_t be a homotopy from f_0 to f_1. It lifts to a homotopy $\tilde{f}_t$ of $\tilde{f}_0$ to an isometry $\tilde{f}_1$ of $\mathbf{H}^3$.

We claim that $\tilde{f}_1$ is independent of the choice of homotopy f_t. Let f'_t be another homotopy from f_0 to f_1. Define $H : M^* \times I \to M^*$ by $H(x,t) = f_{2t}(f_0^{-1}(x))$ for $t \in [0, \frac{1}{2}]$ and $H(x,t) = f'_{2-2t}(f_0^{-1}(x))$ for $t \in [\frac{1}{2}, 1]$. This homotopy satisfies the conditions of Lemma 7.2, and so the loop $H(f_0(x_0^*), t)$ is homotopically trivial. This implies that the paths $f_t(x_0^*)$ and $f'_t(x_0^*)$ are path homotopic, and so their liftings $\tilde{f}_t(\tilde{x}_0)$ and $\tilde{f}_t(\tilde{x}_0)$ have the same endpoint $\tilde{f}_1(\tilde{x}) = \tilde{f}'_1(\tilde{x})$. Thus $\tilde{f}_1 = \tilde{f}'_1$.

Thus the function $\Psi : \pi_1(M, x_0) \to \mathit{Isom}(\mathbf{H}^3)$ given by $\Psi(\alpha) = \tilde{f}_1$ is well defined. Note that if $\alpha \in \operatorname{im} q_*$, then $f_0 = f_1 = id_{M^*}$, and $\tilde{f}_0 = \tilde{f}_1$. Let $\Gamma = \operatorname{im} \Psi$ and $\Gamma_0 = \Psi(\operatorname{im} q_*)$.

We next show that Ψ is a homomorphism. Suppose $\Psi(\beta) = \tilde{g}_1$ and $\Psi(\alpha\beta) = \tilde{h}_1$. Let γ^* be the image of β^* under f_0. Then $\alpha^*\gamma^* = (\alpha\beta)^*$, and so $f_0(g_0(x_0^*)) = f_0(\beta^*(1)) = \gamma^*(1) = h_0(x_0^*)$. Thus $f_0 \circ g_0 = h_0$, and $f_t \circ g_t$ is a homotopy from this map to the isometry $f_1 \circ g_1$. By the uniqueness of h_1 we have that $f_1 \circ g_1 = h_1$. By choosing the homotopy h_t from h_0 to h_1 to be $f_t \circ g_t$, we see that $\tilde{h}_1 = \tilde{f}_1 \circ \tilde{g}_1$, and so $\Psi(\alpha\beta) = \Psi(\alpha)\Psi(\beta)$.

We now show that Ψ is one to one. Suppose $\Psi(\alpha) = \tilde{f}_1 = id_{\mathbf{H}^3}$. If $\alpha \in \operatorname{im} q_*$, then $\tilde{f}_0 = \tilde{f}_1$, and so α is trivial. So assume that this is not the case. Then f_1 is the identity of M^*, and f_0 is not. Since the covering is finite sheeted f_0^n is the identity for some $n > 0$. Define $H : M^* \times I \to M^*$ by $H(x,t) = f_0^{k-1}(f_{k-nt}(x))$ for $t \in [\frac{k-1}{n}, \frac{k}{n}]$. Then $H(x, \frac{k}{n}) = f_0^k$. By Lemma 7.2 the class μ of the loop $m(t) = H(x_0^*, t)$ is trivial in $\pi_1(M^*, x_0^*)$. Let ρ be the path class of the path $r(t) = f_{1-t}(x_0^*)$ joining x_0^* and $f_0(x_0^*)$. Then $q_*(\rho)$ is non-trivial, but $(q_*(\rho))^n = q_*(\mu)$ is trivial. Thus $\pi_1(M, x_0)$ has torsion, contradicting the fact that M is aspherical [10, Corollary 9.9].

Γ is discrete because it contains the finite index discrete subgroup Γ_0 [21, Lemma 8, p. 177]. Thus it acts discontinuously on $\mathbf{H}^3$. It acts freely because otherwise it would have torsion, contradicting the asphericity of M. Hence $N = \mathbf{H}^3/\Gamma$ is a 3-manifold with $\pi_1(N) \cong \pi_1(M)$. It follows from asphericity that N and M are homotopy equivalent, and so N is closed and orientable; hence $\Gamma \subset PSL(2, \mathbf{C})$. □

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